**Numerical Methods for Science and Engineering**

**Lecture Note 2**

**Solution of Linear System**

**2.1 Solution of Linear System by Elimination**

**Equivalent Systems:** Two systems of equations are called equivalent if and only if they have the same solution set.

**Elementary Transformations:** A system of equations is transformed into an equivalent system if the following elementary operations are applied on the system:

(1) two equations are interchanged

(2) an equation is multiplied by a non-zero constant

(3) an equation is replaced by the sum of that equation and a multiple of any other equation.

**Gaussian Elimination**

The process which eliminates an unknown from succeeding equations using elementary operations is known as Gaussian elimination. The equation which is used to eliminate an unknown from the succeeding equations is known as the **pivotal equation**. The coefficient of the eliminated variable in a pivotal equation is known as the **pivot**. If the pivot is zero, it cannot be used to eliminate the variable from the other equations. However, we can continue the elimination process by interchanging the equation with a nonzero pivot.

**Solution of a Linear System**

A systematic procedure for solving a linear system is to reduce a system that is easier to solve. One such system is the echelon form. The **back substitution** is then used to solve the system in reverse order.

A system is in ***echelon form*** or ***upper triangular form*** if

(i) all equations containing nonzero terms are above any equation with zeros only.

(ii) the first nonzero term in every equation occurs to the right of the first nonzero term in the equation above it.

**2.2 Pivotal Elimination Method**

Computers and calculators use fixed number of digits in its calculation, and we may need to round the numbers. This introduces error in the calculations. Also, when two nearly equal numbers are subtracted, the accuracy in the calculation is lost. Consider the following example to study the difficulty faced in the elimination method.

**Example 2.1:**

The linear system





has the exact solution  and .

Solve the above system using four-digit arithmetic i.e. keeping 4 s.f. at each value by

(a) Gaussian elimination without changing the order.

(b) Gaussian elimination by interchanging the order of the equations.

**Solution**

Using four-digit arithmetic the system can be written as

 (E1)

 (E2)

(a) Here a multiplier is .

Multiplying (E1) by 1410 and retaining 4 digits with rounding, we have



Subtracting we get



To 4 digits this can be written as



This gives-



Substituting the value of *y* in (E1), we get





This produces a wrong result.

(b) Interchanging the rows, we can write

 (E3)

 (E4)

For this system multiplier is



The replacement of (E4) by  reduces the system to





The 4-digits arithmetic gives the solution-

 and 

which is the exact solution of the system.

This example shows that the order in which we treat the equations for elimination affects the accuracy in the elimination process.

Note that . Thus, in eliminating *x*, we should use the equation with maximum numerical coefficient of *x*.

**2.2.1 Gaussian Elimination with Partial Pivoting**

**Partial Pivoting**

* In principle, any nonzero value will do as pivot, but in practice pivot should be chosen to minimize error propagation.
* To avoid amplifying previous rounding errors when multiplying remaining portion of

matrix by elementary elimination matrix, multipliers should not exceed 1 in magnitude.

* This can be accomplished by choosing entry of largest magnitude on or below diagonal as pivot at each stage.
* Such **partial pivoting** is essential in practice for numerically stable implementation of Gaussian elimination for general linear systems.

**Complete Pivoting**

* Complete pivoting is more exhaustive strategy in which largest entry in entire remaining unreduced submatrix is permuted into diagonal pivot position.
* Requires interchanging columns as well as rows, leading to factorization:

with unit lower triangular, upper triangular, and and permutations**.**

* Numerical stability of complete pivoting is theoretically superior, but pivot search is more expensive than for partial pivoting.
* Numerical stability of partial pivoting is more than adequate in practice, so it is almost always used in solving linear systems by Gaussian elimination.

**Algorithm of partial pivoting:**

(a) Select pivotal equation for a variable for elimination (the equation with maximum numerical coefficient of that variable in the system).

(b) Eliminate the chosen variable from the remaining equations with respect to the pivotal equation.

(c) Repeat the process for the subsystem.

(d) Using back-substitution find solutions by using pivotal equations.

For convenience of calculations, we made the coefficients of eliminated variable to unity before elimination. This is illustrated through an example below.

**Example 2.3 :** The currents running through an electrical [system](javascript:def('/Glossary/glossaryterm.aspx?word=System',%20500,%20500);) are given by the following [system](javascript:def('/Glossary/glossaryterm.aspx?word=System',%20500,%20500);) of equations.  The three currents, I1, I2, and I3, are measured in amps.

.

a. Solve the above [system](javascript:def('/Glossary/glossaryterm.aspx?word=System',%20500,%20500);) to find the currents in this circuit using Gaussian elimination with partial pivoting.

b. Justify your result by direct substitution in the original equation.

c. Write MATLAB codes to solve by left division (backslash) operator.

**Solution:**  a.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Operations |  |  |  | Constant | Eqn # |  |
|  | 8 | 3 | -5 | 3 | E1 |  |
|  | 10 | 7 | 2 | 4 | E2 |  |
|  | 6 | 4 | 7 | 8 | E3 |  |
| E2/10 | 1.000 | 0.700 | 0.200 | 0.400 | E4 | 1st Piv eqn |
| E1/8 | 1.000 | 0.375 | -0.625 | 0.375 | E5 |  |
| E3/6 | 1.000 | 0.667 | 1.167 | 1.333 | E6 |  |
| E5-E4 | 0 | -0.325 | -0.825 | -0.025 | E7 |  |
| E6-E4 | 0 | -0.033 | 0.967 | 0.933 | E8 |  |
| E7/(-0.325) | 0 | 1 | 2.538 | 0.0769 | E9 | 2nd Piv eqn |
| E8/(-0.033) | 0 | 1 | -29.303 | -28.273 | E10 |  |
| E9-E10 | 0 | 0 | 31.84 | 28.349 | E11 |  |
| E10/(-32.233) | 0 | 0 | 1 | 0.890 |  | 3rd Piv eqn |

The triangular form of the system can be summarized as

Solving by back substitution, we have,

b. Justification:

c. >> A = [8 3 -5; 10 7 2; 6 4 7].

>> b = [3; 4; 8].

>> Solution = A\b

Solution = 1.7500, -2.1829, 0.8902

Partial pivoting is adequate for most of the simultaneous equations which arise in practice. But we may encounter sets of equations where wrong or incorrect solutions may result. To improve the calculation in such cases **total pivoting** is used. In total pivoting, maximum magnitude of the coefficients of the variables is used for the pivot in each elimination.

**2.3 Solution of Linear System by Iterative Method**

To solve a linear system by iteration, we solve each equation for one of the variables, in turn, in terms of the other variables. Starting from an approximation to the solution, if convergent, derive a new sequence of approximations. Repeat the calculations till the required accuracy is obtained. What motivates us to use iterative schemes is the possibility that inverting may be very difficult, to the extent that it may be worthwhile to invert a much ‘easier’ matrix several times, rather than inverting directly only once.

An iterative method converges, for any choice of the first approximation, if every equation satisfies the condition that the magnitude of the coefficient of solving variable is greater than the sum of the absolute values of the coefficients of the other variables. A system satisfying this condition is called diagonally dominant. A linear system can always be reduced to diagonally dominant by elementary operations.

For example, in the system  (E1)  

 (E2) 

 (E3) 

is not diagonally dominant. Rearranging as (E3) − (E1), (E2), (E1), we have

 

 



The system reduces to diagonally dominant form.

**2.3.1. Jacobi Iterative Method:**

In this method, a fixed set of values is used to calculate all the variables and then repeated for the next iteration with the values obtained previously. The iterative formulas of the above system are

The Gauss-Seidel iterative method is more efficient than Jacobi’s iterative method and explained through an example.

**2.3.2. Gauss-Seidel Iterative Method**

In this method, the value of each variable is calculated using the most recent approximations to the values of the other variables. The Gauss-Seidel iterative formulas of the above system are

If initial values are not supplied, we can start with initial values-

and perform the iterations until required accuracy is achieved.

**Example 2.4:** Given the linear system .

a. Reduce the above system to diagonally dominant form.

b. Write the corresponding Gauss-Seidel iteration formula.

c. Compute two iterations to estimate the roots to 2 d.p. with and .

d. Write MATLAB code to iterate the above formula four times.

**Solution:**

1. Eq(2)

Eq(1)-Eq(2)

Eq(2)-Eq(3)

1. Gauss-Seidel formula

1. Starting with initial values

When , we have

For , we have

Solution to 2 d.p. is

d. >> x(1)=1.5;

>> y(1)=-1;

>> z(1)=1;

>> iter(1)=0;

>> for n=1:4

iter(n+1)=n;

x(n+1)=(16+3\*y(n)-2\*z(n))/8;

y(n+1)=(-9+2\*x(n+1)-z(n))/8;

z(n+1)=(1+2\*x(n+1)-4\*y(n+1))/10;

end

>> Solution = [iter',x',y',z']

Solution =

0 1.5000 -1.0000 1.0000

1.0000 1.3750 -0.9063 0.7375

2.0000 1.4758 -0.8482 0.7345

3.0000 1.4983 -0.8422 0.7366

4.0000 1.5000 -0.8421 0.7368

**Exercise-2**

|  |  |
| --- | --- |
| **1.** | A chemist has 3 different acid solutions. The first acid solution contains acid, the second contains and the third contains acid. The chemist wants to use all 3 solutions to obtain a mixture of 100 liters containing acid. The supply of acid is low. Consequently, solution must be used twice as much as solution. How many liters of each solution should be used?  Answer: 10% acid = 25 unit; 40% acid = 25/2 unit; 60% acid = 125/2 unit |

|  |  |
| --- | --- |
| 2. | A movie theater is filled the capacity of The theater charges taka for children, 70 for students and 125 for adults. There are half as many adults as there are students. If the total ticket sales were 24150 takas, how many children, students and adults attended? |

|  |  |  |
| --- | --- | --- |
| 3. |  |  |
|  | **i.** | Solve the above linear system using Gaussian elimination with pivoting (partial/total). |
|  | **ii.** | Justify your result by direct substitution in the original equations. |
|  | **iii.** | Write MATLAB codes to solve by left division (backslash) operator. |

|  |  |  |  |
| --- | --- | --- | --- |
| 4. |  | using    using  using | |
|  | **i.** | Reduce the above system to diagonally dominant form. | |
|  | **ii.** | Write the corresponding Gauss-Seidel and Jacobi iteration formula. | |
|  | **iii.** | Compute two iterations to estimate the roots to 3 d.p. with the given initial values. | |
|  | **iv.** | Justify your result by direct substitution in the original equations. | |
|  | **v.** | Write MATLAB codes to solve by left division (backslash) operator. | |
| 5. |  | Consider the linear system: |
|  | **i.** | Reduce the above system to diagonally dominant form. |
|  | **ii.** | Write the corresponding Gauss-Seidel iteration formula. |
|  | **iii.** | Compute two iterations to estimate the roots to 2 d.p with the following initial values *x =* 2, *y* = 1, *z=* -6. |
|  | **iv.** | Justify your result by direct substitution in the original equations. |
|  | **v.** | Write MATLAB codes to iterate the above formula four times. |

6. Determine the loop currents of the given circuit by solving the linear system using Gaussian elimination with pivoting-

|  |  |
| --- | --- |
| For loop 1:  For loop 2:  For loop 3: | Description: C:\Users\Administrator\Desktop\circuit2.tif |

**7.** Given the linear system:

1. Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p.
2. Reduce the above system to diagonally dominant form and write the corresponding Gauss- Seidel iteration formula.
3. Compute **one** iterations to estimate the solutions to 2 d.p. with
4. Write MATLAB codes to solve by left division (backslash) operator.

**8.** Given the linear system:

1. Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p.
2. Reduce the above system to diagonally dominant form and write the corresponding

Gauss-Seidel iteration formula.

1. Compute one iterations to estimate the solutions to 2 d.p. with
2. Write MATLAB codes to solve by left division (backslash) operator.

**9**. Tracy, Danielle and Sherri bought snacks for a birthday party. They each bought the items shown in the following table at the local convenience store. Find the price of each item.

|  |  |  |  |
| --- | --- | --- | --- |
| Number of bags of potato chips | Number of litres of pop | Number of chocolate bars | Cost  ($) |
| 2 | 9 | 5 | 21.00 |
| 3 | 2 | 10 | 20.88 |
| 8 | 3 | 4 | 13.17 |

i. Construct the system of linear equation from the above problem.

ii. Reduce the system obtained in (i) to diagonally dominant form.

iii. Write the corresponding Gauss-Seidel iteration formula.

iv. Compute two iterations to estimate the roots to 2 d.p using the following initial values

v. Justify your result by direct substitution in the original equations.

vi. Write MATLAB codes to iterate the above formula four times.

**10.** Jesse, Maria and Charles went to the local craft store to purchase supplies for making decorations for the upcoming dance at the high school. Jesse purchased three sheets of craft paper, four boxes of markers and five glue sticks. His bill, before taxes, was $24.40. Maria spent $30.40 when she bought six sheets of craft paper, five boxes of markers and two glue sticks. Charles, purchases totaled $13.40 when he bought three sheets of craft paper, two boxes of markers and one glue stick. Determine the unit cost of each item.

i. Construct the system of linear equation from the above problem.

ii. Solve the system of linear equation using Gaussian elimination with partial pivoting.

iii. Write MATLAB codes to solve by left division (backslash) operator.

|  |  |  |
| --- | --- | --- |
| **11**. | Cory, Josh, and Dan went shopping for Halloween treats. Cory bought 3 chocolate pumpkins, 4 masks and 8 candy witches. He spent $36.65. Josh bought 5 chocolate pumpkins,3 masks and 10 candy witches. He spent $37.50. Dan bought 4 chocolate pumpkins, 5 masks and 6 candy witches. He spent $43.45. and algebraically. | |
|  | **i.** | Write a system of equations to represent this problem. |
|  | **ii.** | Calculate the unit price of each item purchased using Gaussian elimination with pivoting. |
|  | **iii.** | Write MATLAB codes to solve by left division (backslash) operator.  Answer: |

**12.** A local computer company sells three types of laptop computers to three nearby

stores. The number of laptops ordered by each store and the amount owing to the

company for the order is shown in the following table:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Store | Laptop A | | Laptop B | | Laptop C | | Amount Owing ($) | |
| Wal-Mart | | 10 | | 8 | | 6 | | 21 200 |
| Sears | | 7 | | 9 | | 5 | | 18 700 |
| Target | | 8 | | 4 | | 3 | | 13 000 |

i. Write a system of equations to represent the above information.

ii. Solve the system of linear equation using Gaussian elimination with partial pivoting.

iii. Write MATLAB codes to solve by left division (backslash) operator.

|  |  |  |
| --- | --- | --- |
| **13**. | Consider the linear system: | |
|  | **i.** | Reduce the above system to diagonally dominant form. |
|  | **ii.** | Write the corresponding Gauss-Seidel iteration formula. |
|  | **iii.** | Compute two iterations to estimate the roots to 3 d.p. |
|  | **iv.** | Justify your result by direct substitution in the original equations. |
|  | **v.** | Write MATLAB codes to iterate the above formula four times.  Answer: |

**14.** The following system of equations was generated by applying the mesh current law in the circuit shown in the adjacent figure.

2 Ω

**20 V**

40 Ω

35 Ω

**I3**

30 Ω

**I2**

**I1**

45 Ω

1. Write down the system in simplified form.
2. Solve the system of equations using Gauss-Seidel iteration correct to 2 decimal places.

**15.** Given the linear system:

i. Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p.

ii. Reduce the above system to diagonally dominant form and write the corresponding Gauss- Seidel iteration formula.

iii. Compute **one** iterations to estimate the solutions to 2 d.p. with

iv. Write MATLAB codes to solve by left division (backslash) operator.

Answer:

**16.** Given the linear system:

i. Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p.

ii. Reduce the above system to diagonally dominant form and write the corresponding Gauss-Seidel iteration formula.

iii. Compute one iterations to estimate the solutions to 2 d.p. with

iv. Write MATLAB codes to solve by left division (backslash) operator.

Answer:

**17.** An electronics company produces transistors, resistors, and computer chips. Each transistor requires four units of copper, one unit of zinc, and two units of glass. Each resistor requires three, three and one units of the three materials, respectively, and each computer chip requires two, one, and three units of these materials, respectively. Putting this information into table form, we get:

|  |  |  |  |
| --- | --- | --- | --- |
| **Component** | **Copper** | **Zinc** | **Glass** |
| Transistors | 4 | 1 | 2 |
| Resistors | 3 | 3 | 1 |
| Computer chips | 2 | 1 | 3 |

Supplies of these materials vary from week to week, so the company needs to determine a different production run each week. For example, one week the total amounts of materials available are 960 units of copper, 510 units of zinc, and 610 units of glass. Set up the system of equations modeling the production run, and use Excel, MATLAB, or Mathcad, to solve for the number of transistors, resistors, and computer chips to be manufactured this week.

18. Andrea sells photographs at art fairs. She prices the photos according to size: small photos

cost $10, medium photos cost $15, and large photos cost $40. She usually sells as many small

photos as medium and large photos combined. She also sells twice as many medium photos as

large. A booth at the art fair costs $300.

If her sales go as usual, how many of each size photo must she sell to pay for the booth?

19. In the spirit of Christmas and New Years' resolutions, suppose that what we were on a diet

and needed to eat precisely 245 calories, 6 grams of protein, and 7 grams of fat for breakfast.

Unfortunately, I open my cupboard to see that all I have is three boxes of cereal: Cheerios,

Cinnamon Toast Crunch, and Rice Krispies. There nutritional information per serving is as

follows:

|  |  |  |  |
| --- | --- | --- | --- |
| Cereal | Calories | Protein | Fat |
| Cheerios | 120 | 4 | 2 |
| Cinnamon Toast Crunch | 130 | 3 | 5 |
| Rice Krispies | 105 | 1 | 2 |

Now, normally, I would dive in and gorge myself on Cinnamon Toast Crunch†

because they're delicious but, I need to stick to my new year’s resolution.